ISSN: 2277-9655
Impact Factor: 4.116
CODEN: IJESS7

IJESRT

INTERNATIONAL JOURNAL OF ENGINEERING SCIENCES & RESEARCH TECHNOLOGY

SOME FIXED POINT RESULTS IN DISLOCATED QUASI METRIC (DQ-METRIC) SPACES

Dr C Vijender*
* Dept of Mathematics, Sreenidhi Institute of Sciece and Technology, Hyderabad

DOI: 10.5281/zenodo.221003

ABSTRACT
The aim of this paper is to investigate some fixed point results in dislocated quasi metric (dq-metric) spaces. Fixed point results for different types of contractive conditions are established, which generalize, modify and unify some existing fixed point theorems in the literature. Appropriate examples for the usability of the established results are also given. We notice that by using our results some fixed point results in the context of dislocated quasi metric spaces can be deduced.

KEYWORDS: complete dq-metric space, contraction mapping, self-mapping, Cauchy sequence, fixed point.

INTRODUCTION
Fixed point theory is one of the most dynamic research subjects in nonlinear analysis. In this area, the first important and significant result was proved by Banach in 1922 for a contraction mapping in a complete metric space. The well-known Banach contraction theorem may be stated as follows: 'Every contraction mapping of a complete metric space *X* into itself has a unique fixed point' (Bonsall 1962).

Dass and Gupta [1] generalized the Banach contraction principle in a metric space for some rational type contractive conditions.

In the current manuscript, we establish some fixed point results for single and a pair of continuous self-mappings in the context of dislocated quasi metric spaces which generalize, modify and unify the results of Aage and Salunke [2,3], Manvi Kohli[4], Patel and Patel [5], Madhu Shrivastava *et al.* [6] and Zeyada *et al.* [7]. Throughout the paper $\mathbf{R}^+$ represents the set of non-negative real numbers.

PRELIMINARIES
Definition 2.1 ([7])
Let *X* be a non-empty set, and let $d:X\times X\rightarrow R+$ be a function satisfying the following conditions:
(d_1) $d(x, x)=0$;
(d_2) $d(x, y)=d(y, x)=0$ implies that $x=y$;
(d_3) $d(x, y)=d(y, x)$ for all $x, y, z\in X$;
(d_4) $d(x, y)\leq d(x, z)+d(z, y)$ for all $x, y, z\in X$.
If *d* satisfies the conditions from d1 to d4, then it is called a metric on *X*, if *d* satisfies conditions d_2 to d_4, then it is called a dislocated metric (*d*-metric) on *X*, and if *d* satisfies conditions d_2 and d_4, only then it is called a dislocated quasi metric (*dq*-metric) on *X*.
It is evident that every metric on *X* is a dislocated metric on *X*, but the converse is not necessarily true as is clear from the following example.

Example 2.1 Let $X=R^+$ define the distance function $d:X\times X\rightarrow R^+$ by
$d(x, y)=\max\{x, y\}$.
Furthermore, from the following example one can say that a dislocated quasi metric on *X* needs not be a dislocated metric on *X*.

Example 2.2 Let X=[0,1], we define the function $d:X\times X\to R^+$ as
$d(x, y)=|x-y|+|x|$ for all $x, y\in X$.
In our main work we will use the following definitions which can be found in [7].
Definition 2.2 A sequence $\{x_n\}$ in a *dq*-metric space is called a Cauchy sequence if for $\epsilon>0$ there exists a positive integer *N* such that for $m, n\geq N$, we have $d(x_m, x_n)<\epsilon$.
Definition 2.3 A sequence $\{x_n\}$ is called *dq*-convergent in *X* if for $n\geq N$, we have $d(x_n, x)<\epsilon$, where *x* is called the *dq*-limit of the sequence $\{x_n\}$.
Definition 2.4 A *dq*-metric space (X, d) is said to be complete if every Cauchy sequence in *X* converges to a point of *X*.
Definition 2.5 Let (X, d) be a *dq*-metric space, a mapping $T:X\to X$ is called a contraction if there exists $0\leq\alpha<1$ such that
$d(Tx, Ty)\leq\alpha d(x, y)$for all $x, y\in X$ and $\alpha\in[0,1)$.
The following statement is well known (see [7]).
Lemma 1 *Limit in a dq-metric space is unique*.
In [8] Kannan defined a contraction of the following type.
Definition 2.6 Let (X, d) be a metric space, and let $T:X\to X$ be a self-mapping. Then *T* is called a Kannan mapping if
$d(Tx, Ty)\leq\alpha[d(x, Tx)+d(y, Ty)]$for all $x, y\in X$ and $\alpha\in[0,1/2)$. (1)
Kannan [8] established a unique fixed point theorem for a mapping which satisfies condition (1) in metric spaces.
Definition 2.7 ([9])
Let (X, d) be a metric space, a self-mapping $T:X\to X$ is called a generalized contraction if and only for all $x, y\in X$, there exist c_1, c_2, c_3, c_4 such that $\sup\{c_1+c_2+c_3+2c_4\}<1$ and
$d(Tx, Ty)\leq c_1\cdot d(x, y)+c_2\cdot d(x, Tx)+c_3\cdot d(y, Ty)+c_4\cdot[d(x, Ty)+d(y, Tx)]$. (2)
Ciric [9] investigated a unique fixed point theorem for a mapping which satisfies condition (2) in the context of metric spaces.
In the following theorem, Zeyada *et al.* [7] generalized the Banach contraction principle in dislocated quasi metric spaces.
Theorem 2.1 *Let* (X, d) *be a complete dq-metric space*, $T:X\to X$ *be a continuous contraction*, *then T has a unique fixed point in X*.
Aage and Salunke [2] established the following results for single and a pair of continuous mappings in dislocated quasi metric spaces.
Theorem 2.2 *Let* (X, d) *be a complete dq-metric space and* $T:X\to X$ *be a continuous self-mapping satisfying the following condition*:
$d(Tx, Ty)\leq a\cdot d(x, y)+b\cdot d(x, Tx)+c\cdot d(y, Ty)$,
where $a,b,c\geq0$ *with* $a+b+c<1$ *and for all* $x, y\in X$. *Then T has a unique fixed point*.
Theorem 2.3 *Let* (X, d) *be a complete dq-metric space and* $S, T:X\to X$ *be continuous self-mappings satisfying the following condition*:
$d(Sx, Ty)\leq a\cdot d(x, y)+b\cdot d(x, Sx)+c\cdot d(y, Ty)$,
where $a, b, c\geq0$ *with* $a+b+c<1$ *and for all* $x, y\in X$. *Then S and T have a unique common fixed point*.
Furthermore, Aage and Salunke [3] derived the following fixed point theorems with a Kannan-type contraction and a generalized contraction in the setting of dislocated quasi metric spaces, respectively.
Theorem 2.4 *Let* (X, d) *be a complete dq-metric space and* $T:X\to X$ *be a continuous self-mapping satisfying the following condition*:
$d(Tx, Ty)\leq a\cdot[d(x, Tx)+d(y, Ty)]$,
where $a\geq0$ *with* $a<12$ *and for all* $x, y\in X$. *Then T has a unique fixed point*.
Theorem 2.5 *Let* (X, d) *be a complete dq-metric space and* $T:X\to X$ *be a continuous self-mapping satisfying the following condition*:
$d(Tx, Ty)\leq a\cdot d(x, y)+b\cdot d(x, Tx)+c\cdot d(y, Ty)+e\cdot[d(x, Ty)+d(y, Tx)]$,
where $a, b, c, e\geq0$ *with* $a+b+c+2e<1$ *and for all* $x, y\in X$. *Then T has a unique fixed point*.
Isufati [10] derived the following two results, where the first one generalized the result of Dass and Gupta [1] in dislocated quasi metric spaces.
Theorem 2.6 *Let* (X, d) *be a complete dq-metric space and* $T:X\to X$ *be a continuous self-mapping satisfying the following condition*:

$$d(Tx, Ty)\leq a\cdot\frac{d(y, Ty)[1+d(x, Tx)]}{1+d(x, y)}+b\cdot d(x, y),$$

where $a, b>0$ *with* $a+b<1$ *and for all* $x, y\in X$. *Then T has a unique fixed point*.

Theorem 2.7 *Let* (X, d) *be a complete dq-metric space and* $T:X\to X$ *be a continuous self-mapping satisfying the following condition*:

$d(Tx, Ty)\le a\cdot d(x, y)+b\cdot d(y, Tx)+c\cdot d(x, Ty)$,

where a, b, c>0 *with* $\sup\{a+2b+2c\}<1$ *and for all* x, y∈X. *Then T has a unique fixed point*.

In [4] Kohli, Shrivastava and Sharma proved the following theorem in the context of dislocated quasi metric spaces which generalized Theorem 2.6.

Theorem 2.8 *Let* (X, d) *be a complete dq-metric space and* $T:X\to X$ *be a continuous self-mapping satisfying the following condition*:

$$d(Tx, Ty)\le a\cdot\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}+b\cdot d(x, y)+c\cdot d(y, Ty),$$

where a, b, c>0 *with* a+b+c<1 *and for all* x, y∈X. *Then T has a unique fixed point*.

For rational type contraction conditions Madhu Shrivastava *et al.* [6] proved the following theorem in a dislocated quasi metric space.

Theorem 2.9 *Let* (X, d) *be a complete dq-metric space and* $T:X\to X$ *be a continuous self-mapping satisfying the following condition*:

$$d(Tx, Ty)\le a\cdot\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}+b\cdot d(x, y)+c\cdot\frac{d(y,Ty)+d(y,Tx)}{1+d(y,Ty)d(y,Tx)},$$

where a, b, c>0 *with* a+b+c<1 *and for all* x, y∈X. *Then T has a unique fixed point*.

In 2013, Patel and Patel [5] derived the following result in dislocated quasi metric spaces.

Theorem 2.10 *Let* (X, d) *be a complete dq-metric space, and let* $T:X\to X$ *be a continuous self-mapping satisfying the following condition*:

$d(Tx, Ty)\le +c_4\cdot d[(x, Tx)+d(y, Ty)]+c_5\cdot d[d(x, Ty)+d(y, Tx)], c_1\cdot d(x, y)+c_2\cdot d(x, Tx)+c_3\cdot d(y, Ty)$

where $c_1, c_2, c_3, c_4, c_5\ge 0$ *with* $c_1+c_2+c_3+2(c_4+c_5)<1$ *and for all* x, y∈X. *Then T has a unique fixed point*.

MAIN RESULTS

In this section we derive some fixed point theorems with examples for single and a pair of continuous self-mappings in the context of dislocated quasi metric spaces.

Theorem 3.1 *Let* (X, d) *be a complete dq-metric space, and let* $T:X\to X$ *be a continuous self-mapping satisfying the following condition*:

$$d(Tx, Ty)\le a_1\cdot d(x, y)+a_2\cdot d(x, Ty)+a_3\cdot d(y, Tx)+a_4\cdot d(y, Ty)+a_5\cdot\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}+a_6\cdot\frac{d(y,Ty)+d(y,Tx)}{1+d(y,Ty)d(y,Tx)}+a_7\cdot\frac{d(x,Tx)[1+d(y,Tx)]}{1+d(x,y)+d(y,Ty)},\quad (3)$$

where $a_1, a_2, a_3, a_4, a_5, a_6, a_7\ge 0$ *with* $a_1+2(a_2+a_3)+a_4+a_5+3a_6+a_7<1$ *and for all* x, y∈X. *Then T has a unique fixed point in X*.

***Proof* :** Let x_0 be arbitrary in *X*, we define a sequence $\{x_n\}$ by the rule

$x_0, x_1=Tx_0, x_2=Tx_1,\ldots,x_{n+1}=Tx_n$ for all n∈N.

Now we show that $\{x_n\}$ is a Cauchy sequence in *X*. Suppose

$d(x_n, x_{n+1})=d(Tx_{n-1}, Tx_n)$.

By using condition (3) we have

$$d(x_n,x_{n+1})\le a_1\cdot d(x_{n-1}, x_n)+a_2\cdot d(x_{n-1}, x_n)+a_3\cdot d(x_n, x_{n-1})+a_4\cdot d(x_n, x_n)+a_5\cdot\frac{d(x_n,Tx_n)[1+d(x_{n-1},Tx_{n-1})]}{1+d(x_{n-1},x_n)}+a_6\cdot\frac{d(x_n,Tx_n)+d(x_n,Tx_{n-1})}{1+d(x_n,Tx_n)d(x_n,Tx_{n-1})}+a_7\cdot\frac{d(x_{n-1},Tx_{n-1})[1+d(x_n,Tx_{n-1})]}{1+d(x_{n-1},x_n)+d(x_n,Tx_n)}\le a_1\cdot d(x_{n-1}, x_n)+a_2\cdot d(x_{n-1}, Tx_{n+1})+a_3\cdot d(x_n, Tx_n)+a4\cdot d(x_n, Tx_{n+1})+a_5\cdot d(x_n, x_{n+1})+a_6\cdot d(x_n, x_{n+1})+a_6\cdot d(x_n, x_n)+a_7\cdot d(x_{n-1}, x_n),$$

$$d(x_n, x_{n+1})\le\frac{a_1+a_2+a_3+a_6+a_7}{1-(a_2+a_3+a_4+a_5+2a_6)}.d(x_{n-1}, x_n)\cdot$$

Let $h=\dfrac{a_1+a_2+a_3+a_5+a_6+a_7}{1-(a_2+a_3+a_4+2a_6)}$.

Clearly, h<1 because $a_1+2a_2+2a_3+a_4+a_5+3a_6+a_7<1$.
So, $d(x_n, x_{n+1})\leq h\cdot d(x_{n-1}, x_n)$.
Similarly, $d(x_{n-1}, x_n)\leq h\cdot d(x_{n-2}, x_{n-1})$.
Thus, $d(x_n, x_{n+1})\leq h^2\cdot d(x_{n-2}, x_{n-1})$.
Continuing the same procedure, we have
$d(x_n, x_{n+1})\leq h^n d(x_0, x_1)$.
But $0\leq h<1$ so $h^n\to 0$ as $n\to\infty$, which shows that $\{x_n\}$ is a Cauchy sequence in a complete *dq*-metric space. So there exists $z\in X$ such that $x_n\to z$ as $n\to\infty$.
Now we show that *z* is a fixed point of *T*. Since $x_n\to z$ as $n\to\infty$, using the continuity of *T*, we have $\lim_{n\to\infty}Tx_n=Tz$, which implies that $\lim_{n\to\infty}x_{n+1}=Tz$.
Thus Tz=z. Hence *z* is a fixed point of *T*.
Uniqueness. Suppose that *T* has two fixed points *z* and *w* for $z\neq w$. Consider

$$d(z, w)= d(Tz, Tw)\leq a_1\cdot d(z, w)+a_2\cdot d(z, Tw)+a_3\cdot d(w, Tz)+a_4\cdot d(w, Tw)$$
$$+a_5\cdot\frac{d(w, Tw)[1 + d(z, Tz)]}{1+d(z, w)}+a_6\cdot\frac{d(w, Tw) + d(w, Tz)}{1+d(w, Tw)d(w, Tz)}+a_7\cdot\frac{d(z, Tz)[1 + d(w, Tz)]}{1+d(z, w) + d(w, Tw)}. \tag{4}$$

Since *z* and *w* are fixed points of *T*, therefore condition (3) implies that d(z, z)=0 and 0=d(w, w). Finally, from (4) we get

$$d(z, w)\leq(a_1+a_2)\cdot d(z, w)+(a_3+a_6)\cdot d(w, z). \tag{5}$$

Similarly, we have

$$d(w, z)\leq(a_1+a_2)\cdot d(w, z)+(a_3+a_6)\cdot d(z, w). \tag{6}$$

Subtracting (6) from (5) we have

$$\|d(z, w)-d(w, z)\|\leq\|(a_1+a_2)-(a_3+a_6)\|\cdot\|d(z, w)-d(w, z)\|. \tag{7}$$

Since $|(a_1+a_2)-(a_3+a_6)|<1$, so the above inequality (7) is possible if

$$d(z, w)-d(w, z)=0. \tag{8}$$

Taking equations (5), (6) and (8) into account, we have d(z, w)=0 and d(w, z)=0. Thus by (d_2) z=w. Hence *T* has a unique fixed point in *X*.

Theorem 3.2 *Let* (X, d) *be a complete dq-metric space, and let* $S,T:X\to X$ *be two continuous self-mappings satisfying the following condition*:

$$d(Sx, Ty)\leq c_1\cdot d(x, y)+c_2\cdot d(x, Sx)+c_3\cdot d(y, Ty)+c_4\cdot d[(x, Sx)+d(y, Ty)]$$
$$+c_5\cdot d[d(x, Ty)+d(y, Sx)], \tag{9}$$

where $c_1, c_2, c_3, c_4, c_5\geq 0$ *with* $c_1+c_2+c_3+2c_4+4c_5<1$ *and for all* $x, y\in X$. *Then S and T have a unique common fixed point in X*.

Proof Let x_0 be arbitrary in *X*, we define a sequence $\{x_n\}$ by the rule x_0, $x_1=Sx_0,\ldots,x_{2n+1}=Sx_{2n}$ and $x_2=Tx_1,\ldots,x_{2n}=Tx_{2n-1}$ for all $n\in N$. We claim that $\{x_n\}$ is a Cauchy sequence in *X*. For this consider
$d(x_{2n+1}, x_{2n+2})=d(Sx_{2n}, Tx_{2n+1})$.
By using condition (9) we have

$$d(x_{2n+1}, x_{2n+2})\leq c_1\cdot d(x_{2n}, x_{2n+1})+c_2\cdot d(x_{2n}, Sx_{2n})+c_3\cdot d(x_{2n+1}, Tx_{2n+1})+c_4\cdot[d(x_{2n}, Sx_{2n})+d(x_{2n+1}, Tx_{2n+1})]+c_5\cdot[d(x_{2n}, Tx_{2n+1})+d(x_{2n+1}, Tx_{2n})]$$
$$\leq(c_1+c_2+c_4+2c_5)\cdot d(x_{2n}, x_{2n+1})+(c_3+c_4+2c_5)\cdot d(x_{2n+1}, x_{2n+2}).$$

Therefore, finally we have

$$d(x2n+1,x2n+2)\leq \frac{(c_1+c_2+c_4+2c_5)}{1-(c_3+c_4+2c_5)}.d(x_{2n}, x_{2n+1}).$$

Let $h=\dfrac{(c_1+c_2+c_4+2c_5)}{1-(c_3+c_4+2c_5)}$.

Then h<1 as $c_1+c_2+c_3+2c_4+4c_5<1$. Thus $d(x_{2n+1}, x_{2n+2})\leq hd(x_{2n}, x_{2n+1})$ for $n\geq 0$
and $d(x2n,x2n+1)\leq h\cdot d(x2n-1,x2n)$.
So, $d(x_{2n+1}, x_{2n+2})\leq h^2 d(x_{2n-1}, x_{2n})$.
Similarly, we proceed to get
$d(x_n, x_{n+1})\leq h^n d(x_0, x_1)$.
Since $0\leq h<1$ and $n\to\infty$ implies that $h^n\to 0$, which proved that $\{x_n\}$ is a Cauchy sequence in a complete *dq*-metric space. Therefore there exists *z* in *X* such that $x_n\to z$ as $n\to\infty$. Also the sub-sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ converge to *z*. Since *T* is a continuous mapping, therefore
$\lim_{n\to\infty}x_{2n+1}=z\Rightarrow\lim_{n\to\infty}Tx_{2n+1}=Tz\Rightarrow\lim_{n\to\infty}x_{2n+2}=Tz$.

Hence Tz=z.
Similarly, taking the continuity of *S*, we can show that Sz=z.
Hence *z* is the common fixed point of *S* and *T*.
Uniqueness. Suppose that *S* and *T* have two common fixed points *z* and *w* for $z\neq w$. Consider

$$d(z, w)=d(Sz, Tw)\leq c_1\cdot d(z, w)+c_2.d(z, Sz)+c_3\cdot d(w, Tw)$$
$$+c_4\cdot[d(z, Sz)+d(w, Tw)]+c_5\cdot[d(z, Tw)+d(w, Tz)]. \quad (10)$$

Since *z* and *w* are common fixed points of *T* and *S*, condition (9) implies that d(z, z)=0 and d(w, w). Thus equation (10) becomes

$$d(z, w)\leq(c_1+c_5)\cdot d(z, w)+c_5\cdot d(w, z). \quad (11)$$

Similarly,

$$d(w, z)\leq(c_1+c_5)\cdot d(w, z)+c_5\cdot d(z, w). \quad (12)$$

Subtracting (12) from (11) we get

$$\|d(z, w)-d(w, z)\|\leq|c_1|\cdot\|d(z, w)-d(w, z)\|.$$

Since $c_1<1$, so the above inequality is possible if

$$d(z, w)-d(w, z)=0. \quad (13)$$

By combining equations (11), (12) and (13), one can get d(z, w)=0 and d(w, z)=0. Using (d_2) we have z=w. Hence *S* and *T* have a unique common fixed point in *X*.

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